

Dielectric profile variations in high-index-contrast waveguides, coupled mode theory, and perturbation expansions

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Perturbation theory formulation of Maxwell's equations gives a theoretically elegant and computationally efficient way of describing small imperfections and weak interactions in electromagnetic systems. It is generally appreciated that due to the discontinuous field boundary conditions in the systems employing high dielectric contrast profiles standard perturbation formulations fails when applied to the problem of shifted material boundaries. In this paper we developed coupled mode and perturbation theory formulations for treating generic perturbations of a waveguide cross section based on Hamiltonian formulation of Maxwell equations in curvilinear coordinates. We show that our formulation is accurate beyond the first order and rapidly converges to an exact result when used in a coupled mode theory framework even for the high-index-contrast discontinuous dielectric profiles. Among others, our formulation allows for an efficient numerical evaluation of such quantities as deterministic PMD and change in the GV and GVD of a mode due to generic profile distortion in a waveguide of arbitrary cross section. To our knowledge, this is the first time perturbation and coupled mode theories are developed to deal with arbitrary profile variations in high-index-contrast waveguides.

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I. INTRODUCTION

Standard perturbation and coupled mode theory formulations are known to fail or exhibit a very slow convergence [1–6] when applied to the analysis of geometrical variations in the structure of high-index-contrast fibers.

In a coupled mode theory framework, eigenvalues of a matrix of coupling elements approximate the values of the propagation constants of a uniform waveguide of perturbed cross section. When large enough number of modes is included coupled mode theory, in principle, should converge to an exact solution for perturbations of any strength. Perturbation theory is numerically more efficient than coupled mode theory, but it is mostly applicable to the analysis of small perturbations. For stronger perturbations, higher order perturbation corrections must be included converging, in the limit of higher orders, to an exact solution. Both coupled mode and perturbation theories' expansions rely on the knowledge of correct coupling elements.

Conventional approach to evaluation of the coupling elements proceeds by expansion of a solution for the fields in a perturbed waveguide into the modes of an unperturbed system, then computes a correction to the Hamiltonian of a problem due to the perturbation in question and, finally, computes the required coupling elements. Unfortunately, this approach encounters difficulties when applied to the problem of finding perturbed electromagnetic modes in the waveguides with shifted high-index-contrast dielectric boundaries. In particular, expansion of the perturbed modes into an increasing number of the modes of an unperturbed system does not converge to a correct solution when standard form of the coupling elements [7,8] is used. Mathematical reasons of such a failure are still not completely understood

but probably lie either in the *incompleteness* of the basis of eigenmodes of an unperturbed waveguide in the domain of the eigenmodes of a perturbed waveguide, or in the fact that the standard mode orthogonality condition (5) does not constitute a strict norm. We would like to point out that standard coupled mode theory can still be used even in the problem of finding the perturbed modes of a high-index-contrast waveguide with shifted dielectric interfaces, however, expansion basis cannot be chosen of the modes of an unperturbed system. Propagation constants of the perturbed modes are computed by using as an expansion basis eigenmodes of some continuous dielectric profile (empty metallic waveguide, for example, Ref. [6]). Unfortunately, the convergence of such a method with respect to the number of basis modes is very slow (at most linear). Perturbation formulation within this approach is also problematic, thus even for small geometric variations of waveguide profile a full matrix of coupling element has to be computed.

In this paper we introduce a method of evaluating the coupling elements which is valid for any analytical geometrical waveguide profile variations and high-index contrast using the eigenmodes of an unperturbed waveguide as an expansion basis. For the reference, correct first order perturbative expressions in high index-contrast systems for several specialized geometries can be found in Refs. [9–12]. To derive a correct form of the coupling elements in general we first define a convenient way of specifying geometrical variations using a technique of coordinate mapping, and then construct an expansion basis using spatially stretched modes of an unperturbed waveguide. Stretching is performed in such a way as to match the regions of the field discontinuities in the expansion modes with the positions of the perturbed dielectric interfaces. Thus defined expansion modes are designed to have continuous fields in the regions of the continuous dielectric of a perturbed index profile. By substituting such expansions into the Maxwell's equations, we later find the required expansion coefficients. It becomes more

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convenient to perform further algebraic manipulations in a coordinate system where stretched expansion modes become again unperturbed modes of an original waveguide. Thus, the final steps of evaluation of the coupling elements involve transforming and manipulation of Maxwell's equations in the perturbation matched curvilinear coordinates.

In further discussions we formulate uniform geometrical waveguide profile variations in terms of an analytical mapping of an unperturbed dielectric profile onto a perturbed one. Given an unperturbed dielectric profile $\epsilon(x',y')$ in a Euclidean system of coordinates (x',y') (where z is a direction of propagation) we define a mapping $(x'(x,y),y'(x,y))$ such that $\epsilon(x'(x,y),y'(x,y))$ corresponds to an unperturbed profile in a curvilinear coordinate system associated with (x,y,z) . We then perform a coordinate transformation from a Euclidean system of coordinates (x',y',z) into a corresponding curvilinear coordinate system (x,y,z) by rewriting Maxwell's equations in such a coordinate system. Finally, because dielectric profile in a coordinate system (x,y,z) is that of an unperturbed waveguide, we can use the basis set of eigenmodes of an unperturbed Hamiltonian in (x,y,z) coordinates to calculate coupling matrix elements due to geometrical variation of a waveguide profile.

Our paper is organized as following. We first describe some typical geometrical perturbations of fiber profiles. Next, we discuss properties of general curvilinear coordinate transformations and relate them to a particular case of coordinate mappings describing geometrical perturbations of a fiber profile. We then formulate Maxwell's equations in a general curvilinear coordinate system. We apply this formulation to develop the coupled mode and perturbation theories using stretched eigenstates of an unperturbed Hamiltonian as an expansion basis. Next, we conduct a detailed study of the convergence of the coupled mode and perturbation theories for a variety of variations of the geometric profiles in high-index-contrast waveguides. We conclude with analysis of PMD in hollow core Bragg fibers.

In the following, we focus on the geometrical variations in fibers, although geometric variations in generic waveguides [including photonic crystal (PC) waveguides] can be readily described by the same theory (see Ref. [3] for planar waveguides, for example).

II. TYPES OF GEOMETRICAL VARIATIONS OF FIBER PROFILES

We start by considering some common geometrical perturbations of a fiber profile that can arise during manufacturing or service of fibers. Let (x,y) be the coordinates of a Euclidean coordinate system. Consider a general coordinate mapping of the form

$$\begin{aligned} x &= \rho \cos(\theta) + \delta f_x(\rho, \theta), \\ y &= \rho \sin(\theta) + \delta f_y(\rho, \theta), \end{aligned} \quad (1)$$

where $f_{x,y}(\rho, \theta)$ are some analytic functions of variables ρ and θ , and δ controls the strength of a perturbation.

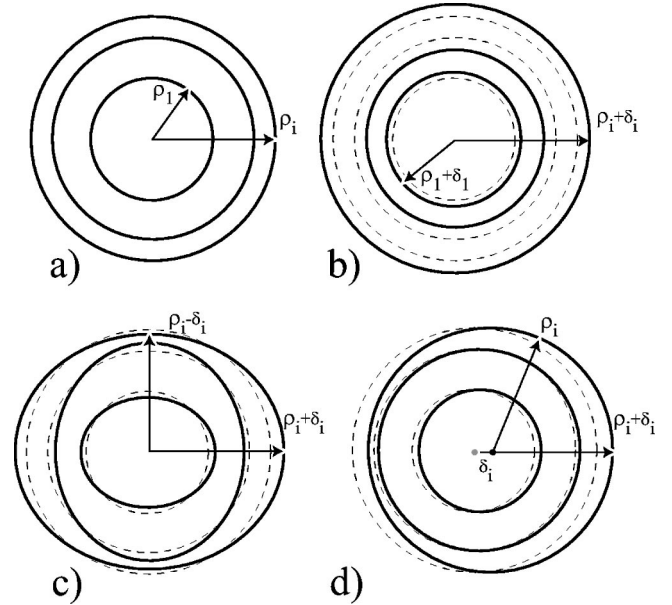


FIG. 1. (a) Dielectric profile of an unperturbed cylindrically symmetrical fiber. Concentric dielectric interfaces are characterized by their radii ρ_i and indexes of refraction n_i . (b) Scaling perturbation. Fiber profile remains cylindrically symmetric, while the radii of the dielectric interfaces become $\rho_i + \delta_i$. (c) Elliptical perturbation. Fiber profile becomes elliptically distorted, large and small radii of the dielectric interfaces become $\rho_i \pm \delta_i$. (d) Example of nonconcentric perturbation. Dielectric interfaces are shifted along the horizontal direction by δ_i .

In unperturbed cylindrically symmetric step index fibers, position of the i th dielectric interface can be described in cylindrical coordinates [mapping Eq. (1) where $\delta=0$] by a set of points, $\rho = \rho_i, \theta \in (0, 2\pi)$ [see Fig. 1(a)].

Typical geometrical imperfections in multilayer fibers include the following.

(1) Scaling perturbations involving cylindrically symmetric variations in the radii of the dielectric layers [see Fig. 1(b)]. If, the radii of the dielectric interfaces are varied from their original values ρ_i by $\delta f(\rho_i)$ [δ is assumed to be small and $f(\rho)$ is assumed to be an analytical function of ρ] then coordinate mapping (1) where $f_x(\rho, \theta) = f(\rho)\cos(\theta)$, $f_y(\rho, \theta) = f(\rho)\sin(\theta)$ will describe such a scaling perturbation. Here, $\theta \in (0, 2\pi)$, $\rho = \rho_i$ define the points on the i th unperturbed dielectric interface, while the corresponding $(x(\rho, \theta), y(\rho, \theta))$ describe the points on the perturbed interfaces.

(2) Elliptical perturbations describing ellipticities of the dielectric interfaces induced in the originally cylindrically symmetric fibers [see Fig. 1(c)]. If, the large and small radii of the i th elliptical interface are $[\rho_i + \delta f(\rho_i)]$ and $[\rho_i - \delta f(\rho_i)]$ respectively, then coordinate mapping (1) where $f_x(\rho, \theta) = f(\rho)\cos(\theta)$, $f_y(\rho, \theta) = -f(\rho)\sin(\theta)$ will describe such an elliptical perturbation.

(3) Nonconcentric perturbations, where nonconcentricity of the dielectric interfaces is induced in the originally cylindrically symmetric fibers [see Fig. 1(d)]. Mapping Eq. (1) where $f_{x,y}(\rho, \theta) = \sum_{n=0}^{+\infty} [F_n^{x,y}(\rho)\cos(n\theta) + G_n^{x,y}(\rho)\sin(n\theta)]$ will describe the most general non-concentric perturbation,

where $Fc_n^{x,y}(\rho)$ and $Fs_n^{x,y}(\rho)$ are some analytic functions of ρ and θ .

III. COUPLED MODE THEORY FOR MAXWELL'S EQUATIONS

In the following, we introduce a Hamiltonian formulation of Maxwell's equations in terms of the transverse field components to address radiation propagation in generic uniform waveguides (also present in Refs. [1,3]). Waveguide is considered to possess translational symmetry in longitudinal \hat{z} direction.

Assuming a standard time dependence of the electromagnetic fields $\mathbf{F}(x,y,z,t) = \mathbf{F}(x,y,z)\exp(-i\omega t)$ (\mathbf{F} denotes elec-

tric or magnetic field vector) and introducing transverse and longitudinal components of the fields $\mathbf{F} = \mathbf{F}_t + \mathbf{F}_z$ with respect to the propagation direction z , Maxwell's equations can be written in terms of the transverse field components as

$$-i \frac{\partial}{\partial z} \hat{B} |\psi\rangle = \hat{A} |\psi\rangle, \quad (2)$$

where operators \hat{A} and \hat{B} are

$$\hat{B} = \begin{pmatrix} 0 & -\hat{z} \times \\ \hat{z} \times & 0 \end{pmatrix}, \quad (3)$$

$$\hat{A} = \begin{pmatrix} \frac{\omega}{c} \epsilon - \frac{c}{\omega} \nabla_t \times \left\{ \hat{z} \left[\frac{1}{\mu} \hat{z} \cdot (\nabla_t \times) \right] \right\} & 0 \\ 0 & \frac{\omega}{c} \mu - \frac{c}{\omega} \nabla_t \times \left\{ \hat{z} \left[\frac{1}{\epsilon} \hat{z} \cdot (\nabla_t \times) \right] \right\} \end{pmatrix}, \quad (3)$$

and Dirac notation is introduced $|\psi\rangle = \begin{pmatrix} \mathbf{E}_t \\ \mathbf{H}_t \end{pmatrix}$. In this form operators on the left and on the right of Eq. (2) are Hermitian.

In the case of a uniform waveguide profile, operator \hat{A} is determined solely by transverse coordinates (x,y) , consequently, the fields—solutions of Eq. (2)—will possess an additional symmetry $\mathbf{F}(x,y,z,t) = \mathbf{F}(x,y)\exp(i\beta z - i\omega t)$. Substitution of these fields into Eq. (2) and denoting an operator of a unperturbed waveguide \hat{A}_0 leads to a generalized Hermitian eigenvalue problem

$$\beta \hat{B} |\psi_\beta^0\rangle = \hat{A}_0 |\psi_\beta^0\rangle. \quad (4)$$

An orthogonality condition between the modes β and β' could then be taken in the form (see Ref. [3])

$$\langle \psi_{\beta^*}^0 | \hat{B} | \psi_{\beta'}^0 \rangle = \frac{\beta'}{|\beta'|} \delta_{\beta,\beta'}, \quad (5)$$

where $\delta_{\beta,\beta'}$ stands for the Kronecker delta. Pure real β 's correspond to the guided modes, pure imaginary β 's correspond to the evanescent modes, and finally, mixed complex β 's correspond to the complex wave modes.

In the current paper we assume a closed uniform waveguide with a linear isotropic lossless and nonmagnetic media so that only the integrable guided and evanescent waves are present in the field expansions. Strictly speaking, this formulation avoids an analysis of coupling to the radiation continuum or a study of leaky modes in the resonant waveguides. In certain regimes, however, current framework still allows an analysis of modes in leaky systems (hollow Bragg fibers and PC waveguides in the example below). Particular methods and their justifications are beyond the scope of this paper and they will only be mentioned in passing. For

the hollow core Bragg fibers and PC waveguides of infinite number of confining layers, for example, radiation leakage in the band gap is zero with modes in the band gap being pure guided and integrable. Consequently, one can use current formulation to find new eigenmodes of a waveguide with perturbed cross section by expanding into a complete set of guided in the band gap, guided in the cladding, and pure evanescent modes. In the case of leaky modes with small radiation loss (originating from the finite number of confining layers in the Bragg or PC fibers, for example) one can still use the modes of an ideal lossless system to approximate the modes of perturbed leaky system in the regions of a waveguide where leaky modes behave like guided modes. In the hollow Bragg and PC fibers this region usually extends from the center of the core to the last confining layer of the Bragg or PC mirrors. In general, introducing absorbing boundary conditions together with conducting boundary conditions outside an absorber recasts the problem of dealing with nonintegrable leaky modes of an open waveguide onto the problem of dealing with lossy (due to absorption and anisotropy of an absorbing dielectric layer) but integrable modes of a closed waveguide. Modification of the current method is possible to include absorption loss and anisotropy of corresponding dielectric materials thus, in principle, allowing the treatment of leaky modes of uniform waveguides including their radiation losses.

When perturbation of a waveguide profile is introduced into a system, operator \hat{A}_0 will be modified. We denote correction to an original operator \hat{A}_0 as $\Delta \hat{A}$. Then, eigenproblem (4) is modified and becomes

$$\tilde{\beta} \hat{B} |\psi\rangle = [\hat{A}_0 + \Delta \hat{A}(z)] |\psi\rangle. \quad (6)$$

One solves Eq. (6) by expanding a solution $|\psi\rangle$ into a linear

combination of some basis functions. If variation corresponding to $\Delta\hat{A}(z)$ is small, then the natural choice of such basis functions would be the eigensolutions $|\psi_\beta^0\rangle$ of an unperturbed operator \hat{A}_0 ,

$$|\psi\rangle = \sum_i C_i |\psi_{\beta_i}^0\rangle, \quad (7)$$

where C_i form a vector of expansion coefficients \vec{C} . Substitution of Eq. (7) into Eq. (6) and further use of the orthogonality condition (5) leads to the following set of coupled linear equations

$$\tilde{\beta}B\vec{C} = (\beta_0B + \Delta A)\vec{C}, \quad (8)$$

where β_0 is a diagonal matrix of propagation constants of the unperturbed modes, normalization matrix B has entries $B_{i,j} = \langle \psi_{\beta_i}^0 | \hat{B} | \psi_{\beta_j}^0 \rangle$, and matrix of coupling elements ΔA has entries $\Delta A_{i,j} = \langle \psi_{\beta_i}^0 | \Delta\hat{A} | \psi_{\beta_j}^0 \rangle$. Moreover, if perturbation of a fiber profile is small, eigenproblem (8) allows perturbative analysis. Mode corresponding to an unperturbed propagation constant β , after perturbation will be described by a modified propagation constant $\tilde{\beta}$, which up to a second order in perturbation strength is given by a standard perturbation theory [13]

$$\tilde{\beta} = \beta + \frac{\langle \psi_{\beta^*} | \Delta\hat{A} | \psi_\beta \rangle}{\langle \psi_{\beta^*} | \hat{B} | \psi_\beta \rangle} + \sum_{\beta' \neq \beta} \frac{\langle \psi_{\beta^*} | \Delta\hat{A} | \psi_{\beta'} \rangle \langle \psi_{\beta'} | \Delta\hat{A} | \psi_\beta \rangle}{\langle \psi_{\beta^*} | \hat{B} | \psi_\beta \rangle \langle \psi_{\beta'} | \hat{B} | \psi_{\beta'} \rangle} \frac{1}{\beta - \beta'}, \quad (9)$$

$$|\psi_{\tilde{\beta}}\rangle = |\psi_\beta\rangle + \sum_{\beta' \neq \beta} \frac{\langle \psi_{\beta'} | \Delta\hat{A} | \psi_\beta \rangle}{\langle \psi_{\beta'} | \hat{B} | \psi_\beta \rangle} \frac{|\psi_{\beta'}\rangle}{\beta - \beta'}. \quad (10)$$

In general, at a particular frequency, the knowledge of β alone is not enough to uniquely characterize an eigenmode of a fiber. Additional labels, such as an angular index of a mode m , indicating the type of angular dependence, are needed. In the case of an elliptical perturbation of a transverse fiber profile, there occurs a split in the doubly degenerate eigenmodes characterized by the same β but having the opposite angular indices m and $-m$. For the case of a doubly degenerate mode, Eq. (9) is not directly applicable. Instead, a split in the corresponding propagation constants is obtained from degenerate perturbation theory [13]. New linearly polarized nondegenerate eigenmodes, which we denote $|\psi_\beta^+\rangle$ and $|\psi_\beta^-\rangle$, up to a phase are found to be

$$|\psi_\beta^\pm\rangle = \frac{1}{\sqrt{2}} (|\psi_{\beta,m}\rangle \pm |\psi_{\beta,-m}\rangle), \quad (11)$$

and the perturbed eigenvalues are

$$\beta^\pm = \beta + \frac{\langle \psi_{\beta^*,m} | \Delta\hat{A} | \psi_{\beta,m} \rangle}{\langle \psi_{\beta^*,m} | \hat{B} | \psi_{\beta,m} \rangle} \pm \left| \frac{\langle \psi_{\beta^*,m} | \Delta\hat{A} | \psi_{\beta,-m} \rangle}{\langle \psi_{\beta^*,m} | \hat{B} | \psi_{\beta,m} \rangle} \right|. \quad (12)$$

Expression (12), in particular, allows to address such an important quantity as a PMD of a fiber. The intermode dispersion parameter τ is defined to be the mismatch of the inverse group velocities of the originally degenerate $m = \pm 1$ modes decoupled by the perturbation. Expressed in terms of the frequency derivative, the intermode dispersion parameter is

$$\tau = \left| \frac{1}{v_g^+} - \frac{1}{v_g^-} \right| = \left| \frac{\partial(\beta^+ - \beta^-)}{\partial\omega} \right| = \left| \frac{\partial\Delta\beta^\pm}{\partial\omega} \right|, \quad (13)$$

where $\Delta\beta^\pm = (\beta^+ - \beta^-)$. The PMD of a doubly degenerate mode exhibiting splitting due to a perturbation is defined to be proportional to τ [14]. A desirable condition of zero PMD at a particular frequency ω then implies a zero value of the frequency derivative of the degeneracy split $\Delta\beta^\pm$, or equivalently $\Delta\beta^\pm$ must be stationary at such a frequency.

Standard methods of evaluation of the coupling matrix elements due to the geometrical distortions of the dielectric interfaces [7,8] employ eigenmodes of an unperturbed waveguide while shifting the dielectric boundaries to extract additional contributions from Maxwell's equations to the original Hamiltonian. While intuitive, these methods are known to fail when dealing with high-index-contrast sharp interfaces (see detailed discussion in Refs. [1,2,4]).

In the following we derive the form of the coupling matrix ΔA of Eq. (8) using transformation of the Maxwell's equations into the curvilinear coordinates that match the geometrical variation in question. This way geometrical variations are interpreted as a change in the geometrical metric of space, while the shapes of dielectric interfaces stay intact, allowing the use of the eigenmodes of an unperturbed waveguide in a corresponding curvilinear coordinate system.

IV. CURVILINEAR COORDINATE SYSTEMS

Following Refs. [4,15,16], we first introduce general properties of the curvilinear coordinate transformations. Let (x^1, x^2, x^3) be the coordinates in a Euclidean coordinate system. We introduce an analytical mapping into a new coordinate system with coordinates (q^1, q^2, q^3) as $[x^1(q^1, q^2, q^3), x^2(q^1, q^2, q^3), x^3(q^1, q^2, q^3)]$. A new coordinate system can be characterized by its covariant basis vectors \vec{a}_i defined in the original Euclidean system as

$$\vec{a}_i = \left(\frac{\partial x^1}{\partial q^i}, \frac{\partial x^2}{\partial q^i}, \frac{\partial x^3}{\partial q^i} \right). \quad (14)$$

Note, if all the coordinates except q^j are kept fixed, then \vec{a}_j is tangential to the set of points described in a Euclidean coordinate system by a curve $[x^1(q^i = \text{const}, q^j, q^k = \text{const}), x^2(q^i = \text{const}, q^j, q^k = \text{const}), x^3(q^i = \text{const}, q^j, q^k = \text{const})]$. We define a reciprocal (contravariant) vector \vec{a}^i as

$$\vec{a}^i = \frac{1}{\sqrt{g}} \vec{a}_j \times \vec{a}_k, \quad (k,j) \neq i, \quad (15)$$

where metric g_{ij} is defined as

$$g_{ij} = \frac{\partial x^k}{\partial q^i} \frac{\partial x^k}{\partial q^j}, \quad (16)$$

and $g = \det(g_{ij})$. From the definition of a reciprocal vector \vec{a}^i , this vector is perpendicular to the surface of constant q^i described in the original Euclidean coordinate system by a set of points $[x^1(q^i = \text{const}, q^j, q^k), x^2(q^i = \text{const}, q^j, q^k), x^3(q^i = \text{const}, q^j, q^k)]$ where $(j \neq k \neq i)$. Vectors \vec{a}_i and their reciprocal \vec{a}^i satisfy the following orthogonality conditions:

$$\vec{a}^i \cdot \vec{a}_j = \delta_{ij}, \quad \vec{a}_i \cdot \vec{a}_j = g_{ij}, \quad \vec{a}^i \cdot \vec{a}^j = g^{ij}, \quad (17)$$

where g^{ij} is an inverse of the metric g_{ij} . In general, a vector may be represented by its covariant components $\vec{E} = e_i \vec{a}^i$ or by its contravariant components $\vec{E} = e^i \vec{a}_i$. These components might have unusual dimensions because the underlying vectors \vec{a}_i and \vec{a}^i are not properly normalized in a Euclidean coordinate system. Components having the usual dimensions are defined by $E_i = e_i / \sqrt{g^{ii}}$, $E^i = e^i / \sqrt{g_{ii}}$ and $\vec{E} = e_i \vec{a}^i = E_i \vec{i}^i$, $\vec{E} = e^i \vec{a}_i = E^i \vec{i}_i$, where \vec{i}_i, \vec{i}^i are unitary vectors. Normalized covariant and contravariant components are connected by $E_i = G_{ij} E^j$, $E^i = G^{ij} E_j$ where $G_{ij} = \sqrt{g^{ii} / g_{jj} g_{ij}}$, $G^{ij} = \sqrt{g_{ii} / g^{jj} g^{ij}}$.

In the forthcoming presentation we concentrate on the analysis of the geometrical fiber profile perturbations that can be described by a general coordinate transformation of the form

$$x(\rho, \theta), y(\rho, \theta), z, \quad (18)$$

where (x, y, z) are the coordinates in a Euclidean coordinate system. In an unperturbed cylindrically symmetric fiber, the i th dielectric interface can be characterized by its radius ρ_i and a corresponding set of points $[x = \rho \cos(\theta), y = \rho \sin(\theta)]$, where $\rho = \rho_i, \theta \in (0, 2\pi)$. Given mapping (18), perturbed i th dielectric interface can still be characterized by its unperturbed radius ρ_i and is represented by a set of points $[x(\rho, \theta), y(\rho, \theta)]$, $\rho = \rho_i, \theta \in (0, 2\pi)$. As demonstrated in Ref. [4], vectors forming local covariant and contravariant coordinate systems associated with coordinates (ρ, θ, z) will have the following properties on the dielectric interfaces. Contravariant basis vectors: \vec{i}^ρ is strictly perpendicular to the perturbed dielectric interface, \vec{i}^θ is almost parallel to the perturbed dielectric with an angle between \vec{i}^θ and tangential to the dielectric interface being proportional to the perturbation strength δ . Covariant basis vectors: \vec{i}_ρ is almost perpendicular to the perturbed dielectric interface with an angle between

\vec{i}_ρ and normal to the dielectric interface being proportional to the perturbation strength δ , while \vec{i}_θ is strictly parallel to the perturbed dielectric interface.

To summarize, geometric perturbations of a fiber profile can be characterized by a small parameter (δ in the examples above) that defines a curvilinear coordinate mapping of an unperturbed profile expressed in coordinates (ρ, θ, z) onto a perturbed profile expressed in Euclidean coordinates $[x(\rho, \theta), y(\rho, \theta), z]$. This mapping defines a curvilinear and, generally, a nonorthogonal coordinate system of coordinates (ρ, θ, z) . The smaller is the profile perturbation the closer is the corresponding curvilinear coordinate system to an original orthogonal system.

V. MAXWELL'S EQUATIONS IN CURVILINEAR COORDINATES

We now present a formulation of Maxwell's equations in general curvilinear coordinates. A well known form of Maxwell's equations in curvilinear coordinates can be found in a variety of references [15–18]. These expressions are compactly expressed in terms of the normalized covariant and contravariant components of the fields and in the absence of free electric currents they are

$$\begin{aligned} \epsilon(q^1, q^2, q^3) \frac{\partial}{\partial ct} \frac{E^i}{\sqrt{g_{ii}}} &= \frac{1}{\sqrt{g}} e^{ijk} \frac{\partial}{\partial q^j} \frac{H_k}{\sqrt{g^{kk}}}, \\ -\mu(q^1, q^2, q^3) \frac{\partial}{\partial ct} \frac{H^i}{\sqrt{g^{ii}}} &= \frac{1}{\sqrt{g}} e^{ijk} \frac{\partial}{\partial q^j} \frac{E_k}{\sqrt{g^{kk}}}, \end{aligned} \quad (19)$$

where e^{ijk} is a Levi-Civita symbol. For further derivations it is most convenient to write Maxwell's equations in terms of the normalized covariant components only. This will allow us later in the paper to express the coupling elements only in terms of the integrals over the fields, avoiding the integrals over the field derivatives. In further derivations we will deal with uniform in z coordinate transformations of the form $[x(\rho, \theta), y(\rho, \theta), z]$ (nonuniform in z profiles are considered in Refs. [4,5]), thus, some of the elements of the metric are trivial $g^{zz} = g_{zz} = 1$, $g^{\rho z} = g^{\theta z} = g_{\rho z} = g_{\theta z} = 0$.

For a uniform fiber, propagation constant is a conserved number. This translational symmetry can be expressed as

$$\begin{pmatrix} \mathbf{E}(\rho, \theta, t) \\ \mathbf{H}(\rho, \theta, t) \end{pmatrix} = \begin{pmatrix} \mathbf{E}(\rho, \theta) \\ \mathbf{H}(\rho, \theta) \end{pmatrix}_{\vec{\beta}} \exp(i\vec{\beta}z - i\omega t). \quad (20)$$

Assuming nonmagnetic materials $\mu = 1$ we express Maxwell's equations (19) in terms of the normalized covariant coordinates only,

$$\begin{aligned}
 \tilde{\beta}H_\theta &= -\sqrt{g^{\theta\theta}} \frac{\partial iH_z}{\partial \theta} + \frac{\omega}{c} \epsilon \sqrt{g} (\sqrt{g^{\rho\rho}g^{\theta\theta}}E_\rho + g^{\rho\theta}E_\theta), \\
 -\tilde{\beta}H_\rho &= \sqrt{g^{\rho\rho}} \frac{\partial iH_z}{\partial \rho} + \frac{\omega}{c} \epsilon \sqrt{g} (g^{\theta\rho}E_\rho + \sqrt{g^{\rho\rho}g^{\theta\theta}}E_\theta), \\
 -\tilde{\beta}E_\theta &= \sqrt{g^{\theta\theta}} \frac{\partial iE_z}{\partial \theta} + \frac{\omega}{c} \sqrt{g} (\sqrt{g^{\rho\rho}g^{\theta\theta}}H_\rho + g^{\rho\theta}H_\theta), \\
 \tilde{\beta}E_\rho &= -\sqrt{g^{\rho\rho}} \frac{\partial iE_z}{\partial \rho} + \frac{\omega}{c} \sqrt{g} (g^{\theta\rho}H_\rho + \sqrt{g^{\rho\rho}g^{\theta\theta}}H_\theta),
 \end{aligned} \tag{21}$$

$$\begin{aligned}
 iE_z &= -\frac{c}{\omega \epsilon \sqrt{g}} \left(\frac{\partial \frac{H_\theta}{\sqrt{g^{\theta\theta}}}}{\partial \rho} - \frac{\partial \frac{H_\rho}{\sqrt{g^{\rho\rho}}}}{\partial \theta} \right), \\
 iH_z &= \frac{c}{\omega \sqrt{g}} \left(\frac{\partial \frac{E_\theta}{\sqrt{g^{\theta\theta}}}}{\partial \rho} - \frac{\partial \frac{E_\rho}{\sqrt{g^{\rho\rho}}}}{\partial \theta} \right).
 \end{aligned} \tag{22}$$

Note that by substituting z components of the electric and magnetic fields (22) into (21) we can form the nonuniform Hamiltonian formulation of the Maxwell's equations in the curvilinear coordinates in terms of the differential equation relating the covariant components of transverse fields only.

For an unperturbed cylindrically symmetric uniform fiber, we denote the eigenfields corresponding to a propagation constant β and an angular integer index m as

$$\begin{pmatrix} \mathbf{E}^0(\rho, \theta, z) \\ \mathbf{H}^0(\rho, \theta, z) \end{pmatrix}_\beta^m = \begin{pmatrix} \mathbf{E}^0(\rho) \\ \mathbf{H}^0(\rho) \end{pmatrix}_\beta^m \exp(i\beta z + im\theta). \tag{23}$$

Fields (23) satisfy equations (21) and (22) where an unperturbed polar coordinate metric is diagonal $g^{0\rho\rho}=1$; $g^{0\rho\theta}=0$; $g^{0\theta\theta}=1/\rho^2$; $g^0=\rho^2$.

VI. COUPLED MODE THEORY FOR MAXWELL'S EQUATIONS IN CURVILINEAR COORDINATES

In the previous sections we introduced geometrical variations of waveguide cross section in terms of the analytical

mapping of an unperturbed onto a perturbed waveguide profile. Next, we construct an expansion basis using the eigenfields of an unperturbed fiber by spatially stretching them in such a way as to match the regions of discontinuity of their field components with the position of the perturbed dielectric interfaces. Finally, we find expansion coefficients by satisfying Maxwell's equations (21) and (22). In the following, we first define an expansion basis and then demonstrate how perturbation theory and a coupled mode theory can be formulated in such a basis.

A. Expansion basis

We demonstrated previously [1,3,4] that eigenmodes of an unperturbed waveguide can be described in the Hermitian Hamiltonian formulation in terms of the transverse fields satisfying an orthogonality condition (5). In the following we will concentrate on the case of fibers, while analogous derivations can be applied to any type of waveguides. For cylindrically symmetrical fibers the natural choice of coordinates are the polar coordinates. Condition of orthogonality (5) for the eigenfields of an unperturbed fiber can be expanded as

$$\begin{aligned}
 \langle \psi_{\beta^*, m}^0 | \hat{B} | \psi_{\beta', m'}^0 \rangle &= \frac{\beta'}{|\beta'|} \delta_{\beta, \beta'} = \int [(H_{\theta\beta^*}^{0m})^* E_{\rho\beta'}^{0m'} \\
 &\quad - (H_{\rho\beta^*}^{0m})^* E_{\theta\beta'}^{0m'} + H_{\theta\beta'}^{0m'} (E_{\rho\beta^*}^{0m})^* \\
 &\quad - H_{\rho\beta'}^{0m'} (E_{\theta\beta^*}^{0m})^*] J(\rho) d\rho d\theta,
 \end{aligned} \tag{24}$$

where in polar coordinates Jacobian of transformation is $J(\rho)=\rho$, and notation for a field component $j \in (\rho, \theta)$ (either electric or magnetic) is $F_{j\beta}^{0m} = F_j^0(\rho) \exp(im\theta)$.

Now, consider a waveguide with a geometry slightly different from an original one. If (x, y, z) is a Euclidean coordinate system associated with a perturbed waveguide, and (ρ, θ, z) is a coordinate system corresponding to an unperturbed cylindrically symmetric fiber, we define a coordinate transformation relating such two coordinate systems as $[x(\rho, \theta), y(\rho, \theta), z]$, and a conjugate coordinate transformation as $[\rho(x, y), \theta(x, y), z]$. Using solutions for the fields of an unperturbed fiber expressed in the coordinates (ρ, θ, z) we form an expansion basis in a Euclidean coordinate system (x, y, z) as

$$|\Psi_{\beta, m}\rangle = \begin{pmatrix} \sqrt{\frac{g^{\rho\rho}}{g^{0\rho\rho}}} E_\rho^0(\rho(x, y)) \tilde{i}^\rho + \sqrt{\frac{g^{\theta\theta}}{g^{0\theta\theta}}} E_\theta^0(\rho(x, y)) \tilde{i}^\theta \\ \sqrt{\frac{g^{\rho\rho}}{g^{0\rho\rho}}} H_\rho^0(\rho(x, y)) \tilde{i}^\rho + \sqrt{\frac{g^{\theta\theta}}{g^{0\theta\theta}}} H_\theta^0(\rho(x, y)) \tilde{i}^\theta \end{pmatrix}_\beta^m \exp[im\theta(x, y)]. \tag{25}$$

There are several important properties that basis vector fields (25) possess. First, if coordinate transformation is orthogonal ($\vec{i}^\rho = \vec{i}_\rho$, $\vec{i}^\theta = \vec{i}_\theta$) then one can show that any two basis fields of (25) are orthogonal in a sense of the orthogonality condition (24), $\langle \Psi_{\beta^*,m} | \hat{B} | \Psi_{\beta',m'} \rangle = (\beta' / |\beta'|) \delta_{\beta,\beta'} \delta_{m,m'}$. Moreover, in the case of nonorthogonal transformations, basis fields (25) will be almost orthogonal with an amount of nonorthogonality proportional to the strength of perturbation.

Second, basis fields (25) are continuous in the regions of continuous dielectric of a perturbed fiber profile. In other words, regions of discontinuity in the field components of the basis fields (25), by construction, coincide with the positions of the perturbed dielectric interfaces.

Finally, due to the properties of the generalized coordinate transformations discussed in the preceding section, \vec{i}^ρ will be strictly normal to the perturbed dielectric interfaces, while \vec{i}^θ will be either strictly tangential (orthogonal coordinate transformation) or almost tangential to the perturbed dielectric interfaces, deviating from strict tangentiality by an amount proportional to the strength of perturbation. As the unperturbed components of the fields E_ρ^0, H_ρ^0 and E_θ^0, H_θ^0 satisfy the correct boundary conditions on the unperturbed dielectric interfaces the field components (25) of (25) basis functions will either exactly satisfy the proper boundary conditions on the perturbed interfaces (orthogonal coordinate transformation) or will satisfy them approximately deviating only by an amount proportional to the strength of perturbation.

B. Coupled mode theory

In the preceding section we derived the forms (21) and (22) of Maxwell's equations in the curvilinear coordinates (ρ, θ, z) . While seemingly complicated, these equations in-

volve an unperturbed, cylindrically symmetric uniform in z dielectric profile $\epsilon(\rho, \theta)$. We look for a solution of Maxwell's equations (21), (22) in terms of the field expansion coefficients $C_{m',\bar{\beta}}^{\beta'}$ of the basis fields (25) which in (ρ, θ, z) coordinate system are essentially the eigenfields of an unperturbed fiber. Thus, in covariant coordinates

$$\begin{pmatrix} E_\rho \\ E_\theta \\ H_\rho \\ H_\theta \end{pmatrix}_{\bar{\beta}} = \sum_{\beta', m'} C_{m',\bar{\beta}}^{\beta'} \begin{pmatrix} \sqrt{\frac{g^{\rho\rho}}{g^{0\rho\rho}}} E_\rho^0(\rho) \\ \sqrt{\frac{g^{\theta\theta}}{g^{0\theta\theta}}} E_\theta^0(\rho) \\ \sqrt{\frac{g^{\rho\rho}}{g^{0\rho\rho}}} H_\rho^0(\rho) \\ \sqrt{\frac{g^{\theta\theta}}{g^{0\theta\theta}}} H_\theta^0(\rho) \end{pmatrix}_{\beta'} \exp(im'\theta). \quad (26)$$

We then substitute expansion (26) into Eq. (21), multiply the modified equations (21) on the left by the proper components of the unperturbed fields and integrate over the fiber cross section to exploit an orthogonality condition satisfied by the unperturbed solutions (24). Manipulating the resultant equations (see Ref. [4] for details) we arrive at the following set of equations:

$$\tilde{\beta} B \tilde{C}_{\bar{\beta}} = M \tilde{C}_{\bar{\beta}}, \quad (27)$$

where $B_{\beta^*,\beta'} = (\beta' / |\beta'|) \delta_{\beta,\beta'}$ is a normalization matrix (5), M is a matrix of coupling elements given by

$$M_{\beta^*,m;\beta',m'} = \langle \psi_{\beta^*,m} | \hat{M} | \psi_{\beta',m'} \rangle$$

$$= \frac{\omega}{c} \int \exp[i(m'-m)\theta] \begin{pmatrix} E_\rho^0(\rho) \\ E_\theta^0(\rho) \\ E_z^0(\rho) \\ H_\rho^0(\rho) \\ H_\theta^0(\rho) \\ H_z^0(\rho) \end{pmatrix}_{\beta^*} \begin{pmatrix} \epsilon d_{\rho\rho} & \epsilon d_{\rho\theta} & 0 & 0 & 0 & 0 \\ \epsilon d_{\theta\rho} & \epsilon d_{\theta\theta} & 0 & 0 & 0 & 0 \\ 0 & 0 & \epsilon d_{zz} & 0 & 0 & 0 \\ 0 & 0 & 0 & d_{\rho\rho} & d_{\rho\theta} & 0 \\ 0 & 0 & 0 & d_{\theta\rho} & d_{\theta\theta} & 0 \\ 0 & 0 & 0 & 0 & 0 & d_{zz} \end{pmatrix} \begin{pmatrix} E_\rho^0(\rho) \\ E_\theta^0(\rho) \\ E_z^0(\rho) \\ H_\rho^0(\rho) \\ H_\theta^0(\rho) \\ H_z^0(\rho) \end{pmatrix}_{\beta'} J(\rho) d\rho d\theta, \quad (28)$$

integration is performed over an unperturbed fiber profile, and nonzero elements of 6×6 matrix \hat{M} are

$$d_{\rho\rho} = g^{\rho\rho} \sqrt{\frac{g g^{0\theta\theta}}{g^{0\rho\rho}}}, \quad (29)$$

$$d_{\rho\theta} = d_{\theta\rho} = g^{\rho\theta} \sqrt{g},$$

$$d_{\theta\theta} = g^{\theta\theta} \sqrt{\frac{g g^{0\rho\rho}}{g^{0\theta\theta}}},$$

$$d_{zz} = -g^0 \sqrt{\frac{g^{0\rho\rho} g^{0\theta\theta}}{g}}.$$

Equations (26) present an eigenproblem with respect to the values of the modified propagation constants $\tilde{\beta}$ and corresponding vectors of expansion coefficients $C_{m,\tilde{\beta}}^\beta$ and can be further solved numerically.

Note that the matrix of coupling elements M is Hermitian and, thus $\langle \psi_{\beta^*,m} | \hat{M} | \psi_{\beta',m'} \rangle = \langle \psi_{\beta',m'} | \hat{M} | \psi_{\beta^*,m} \rangle^*$. In Eq. (28) we can further perform an integration of matrix \hat{M} over θ , $\int \hat{M} \exp[i(m' - m)\theta] d\theta$, as such an integration is decoupled from the integration over the fields which are the functions of ρ only.

To summarize, given a geometric perturbation of a uniform fiber profile, we characterize such a perturbation in terms of a coordinate transformation $[x(\rho, \theta), y(\rho, \theta), z]$ where an original cylindrically symmetric fiber profile in coordinates (ρ, θ, z) is mapped onto a perturbed fiber profile in Euclidean coordinates (x, y, z) . Maxwell's equations transformed into a perturbation defined curvilinear coordinate system can be recast into a generalized Hermitian eigenvalue problem (27). Solving such an eigenvalue problem governed by the matrix of coupling elements (28) and (29) one can find the modes of a perturbed uniform waveguide to any degree of accuracy.

Moreover, Eq. (27) allows a perturbative interpretation. As a metric of a slightly perturbed coordinate system is only slightly different from the metric of an unperturbed coordinate system, that will naturally introduce a small parameter for small geometrical perturbations of a fiber profile. Exactly the same perturbative expansion for the values of the modal propagation constants as in Eqs. (9) and (12) holds in our case, where instead of operator ΔA one would use an operator $M - \beta_0 B$, where β_0 is a diagonal matrix of propagation constants of unperturbed modes.

VII. STUDY OF CONVERGENCE OF THE PERTURBATION EXPANSIONS AND A COUPLED MODE THEORY

We test our theory on a set of geometrical perturbations in the multilayered dielectric waveguides. Modes and propagation constants of unperturbed cylindrically symmetric dielectric waveguides are calculated by transfer matrix approach detailed in Refs. [1,19].

As a test system we consider a two core dielectric waveguide [see Fig. 1(a)] with radii $R_{c1} = 1a$, $R_{c2} = 2a$, and corresponding indices of refraction $n_{c1} = 3$, $n_{c2} = 2$. Dielectric cladding is assumed to be $n_{clad} = 1$. Waveguide is surrounded by a metal jacket (metal is assumed to be ideal) at $R_m = 4a$. Frequency is fixed and equals $\omega = 0.2(2\pi c/a)$, propagation constants are measured in units $2\pi/a$, where a defines the scale and can be chosen at will. Metallic boundary conditions of a waveguide's outer jacket eliminates the radiation continuum allowing a discrete spectrum of guided (propagation constant is pure real) and evanescent modes (propagation constant is complex). There are altogether eight guided modes (forward and backward propagating) of angular index $m=0$, 16 guided modes of angular index m

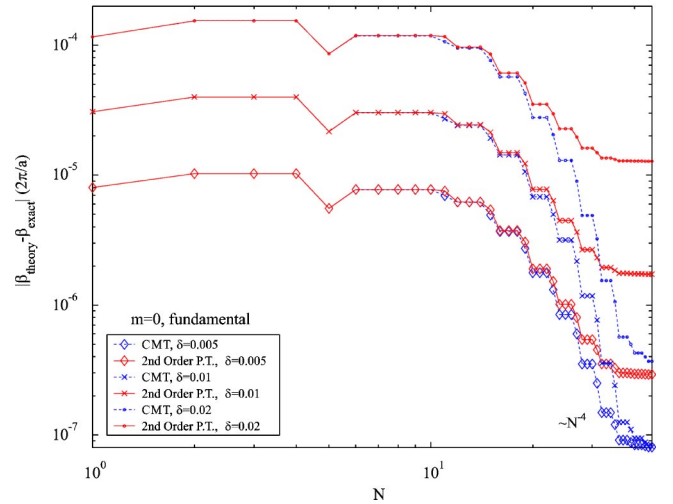


FIG. 2. Error in the values of the propagation constant of the fundamental $m=0$ mode of a perturbed dual core metal jacketed fiber [see Fig. 1(b)] as a function of the number of expansion modes. Dotted curves correspond to the coupled mode theory (CMT) while solid curves correspond to the second order perturbation theory. Strengths of the perturbations are $\delta=0.5\%, 1\%, 2\%$.

$= \pm 1$, 12 guided modes of angular index $m = \pm 2$, 8 guides modes of angular index $m = \pm 3$, and there are no other guided modes for the higher angular indices in this fiber (although there are infinitely many evanescent modes for any angular index m).

First, we analyze convergence of the coupled mode theory for nonuniform scaling perturbations described by coordinate mapping (1)-(1) where waveguide's second dielectric interface and outer metal jacket are kept fixed while waveguide's first core radius is changed from R_{c1} to $R_{c1}(1 + \delta)$. In principle, there are infinitely many mappings that keep the second dielectric interface and an outer metallic jacket intact while changing the radius of the first core. The choice of a particular mapping can influence convergence of the coupled mode theory, however, exactly the same result should be achieved for the equivalent mappings (mappings are equivalent if the positions of the resultant dielectric interfaces are the same). Thus, we choose mapping (1)-(1) in the form

$$f(\rho) = \rho \frac{(\rho - R_{c2})(\rho - R_m)}{(R_{c1} - R_{c2})(R_{c1} - R_m)}.$$

Such a mapping leaves the second dielectric interface and metallic jacket unchanged, while changing the radius of the first core from R_{c1} to $R_{c1}(1 + \delta)$. Coupling elements are calculated exactly from (28) with selection rule $\Delta m = 0$ and \bar{M} matrix given by $d_{\rho\rho} = -\delta\{[\rho f'(\rho) - f(\rho)]/\rho[1 + \delta f'(\rho)]\}$, $d_{\rho\theta} = d_{\theta\rho} = 0$, $d_{\theta\theta} = \delta\{[\rho f'(\rho) - f(\rho)]/\rho + \delta f(\rho)\}$, $d_{zz} = \{1 - \rho/[\rho + \delta f(\rho)][1 + \delta f'(\rho)]\}$. For $\delta = 0.5\%, 1\%, 2\%$ perturbations we plot errors in the propagation constant of perturbed $m=0$ fundamental mode calculated by the coupled mode theory, Eqs. (27)–(29), and second order perturbation theory, Eqs. (9), (28), and (29) as a function of the number of expansion modes N (see Fig. 2). Note, that only the modes with the same angular index m are

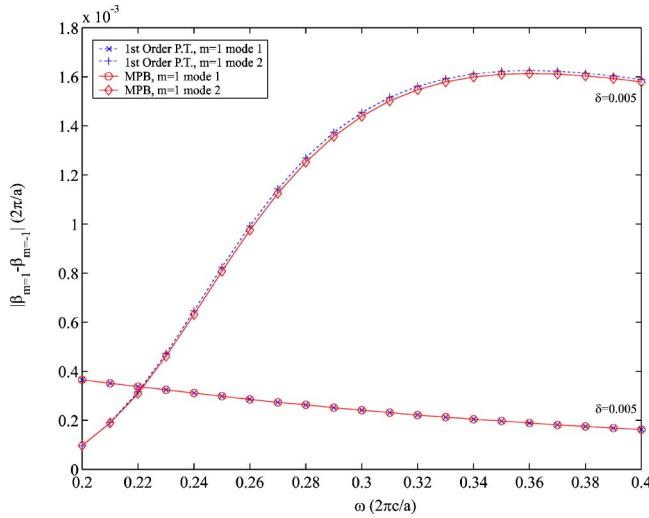


FIG. 3. First order in δ split $|\beta_{m=1} - \beta_{m=-1}|$ in the propagation constants of the first two $m=1$ modes (fundamental, and the closest to the fundamental modes) of a composite core fiber under a non-uniform elliptical perturbation (see Fig. 1c). Solid curves correspond to the frequency domain MPB code, while dotted curves correspond to the first order perturbation theory. Strength of the perturbation is $\delta=0.5\%$ for both modes.

coupled by scaling perturbation. Inclusion of only a single unperturbed fundamental mode $N=1$ into coupled mode theory corresponds to the first order perturbation theory (9). Note from Fig. 2 that the resulting error scales as δ^2 , signifying that the first order perturbation theory is correct, reducing the error to the second order. Inclusion of the rest of the modes (guided and evanescent) leads to a very fast $\sim N^{-2.5}$ convergence of the coupled mode theory (dotted curves). Errors in the second order perturbation theory (solid curves) saturate with only 20–30 expansion modes and the resultant errors scale as δ^3 , signifying that the second order perturbation theory is correct, reducing the error to the third order.

Next, we analyze nonuniform elliptical perturbation of a composite core fiber [see Fig. 1(b)]. We choose mapping (1)–(2) in the form

$$f(\rho) = \rho \frac{(\rho - R_{c2})(\rho - R_m)}{(R_{c1} - R_{c2})(R_{c1} - R_m)}.$$

Such a mapping leaves the second dielectric interface and metallic jacket unchanged, while letting the radius of the first core vary in the range $R_{c1}(1 \pm \delta)$. Coupling elements are calculated to the first order by Eqs. (28) and (29) with selection rule $|\Delta m|=2$ and \bar{M} matrix given by $d_{\rho\rho} = -\delta\{[\rho f'(\rho) + f(\rho)]/2\rho\}$,

$$d_{\rho\theta} = d_{\theta\rho} = \text{sgn}(\Delta m) i \delta \frac{[f(\rho) + \rho f'(\rho)]}{2\rho},$$

$d_{\theta\theta} = \delta\{[f(\rho) + \rho f'(\rho)]/2\rho\}$, $d_{zz} = \delta\{[\rho f'(\rho) - f(\rho)]/2\rho\}$. For $\delta=0.5\%$ ellipticity we plot (Fig. 3) splitting in the propagation constants $|\beta_{m=1} - \beta_{m=-1}|$ of originally degenerate $m = \pm 1$ modes calculated by first order perturbation theory (12) using coupling elements (28). We compare our

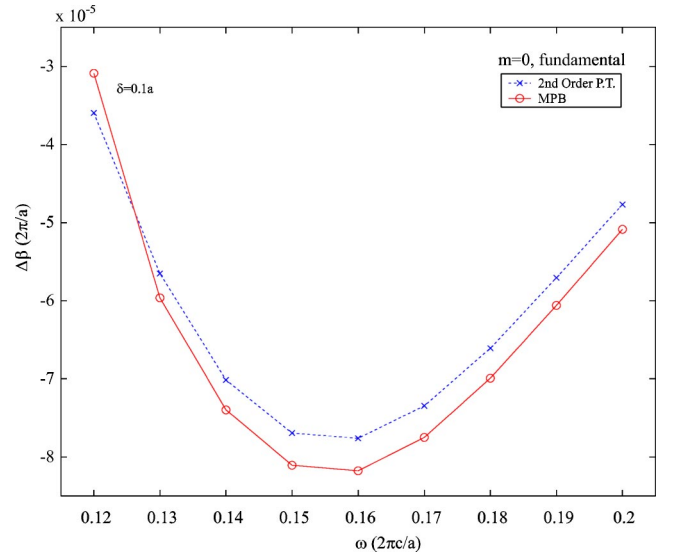


FIG. 4. Second order in δ shift $\Delta\beta$ in the propagation constants of the $m=0$ mode of a composite core fiber under a nonuniform non-circular perturbation [see Fig. 1(d)]. Solid curves correspond to the frequency domain MPB code, while dotted curves correspond to the second order perturbation theory. Strength of the perturbation is $\delta=0.1a$.

theoretical predictions with the results of a frequency domain code called MIT photonic bands (MPB [20]) developed at MIT. Over a large range of frequencies we have an excellent agreement between the first-order perturbation theory results and the results of the frequency domain MPB code.

Finally, we present analysis of nonconcentric perturbation of a composite core fiber where one of the interfaces is shifted with respect to the center of cylindrical symmetry [see Fig. 1(d)]. We choose mapping (1)–(3) in the form

$$f_x(\rho, \theta) = \frac{(\rho - R_{c2})(\rho - R_m)}{(R_{c1} - R_{c2})(R_{c1} - R_m)},$$

$f_y(\rho, \theta) = 0$. Such a mapping leaves the second dielectric interface and metallic jacket unchanged, while shifting the first dielectric interface horizontally by δ . We evaluate change in the propagation constant of the fundamental $m=0$ mode. From expression (9) and the form of the matrix elements (28) one can derive that $M_{\beta^*,0;\beta,0} = 0$, and, hence, the change in the propagation constant of $m=0$ mode scales as second power of perturbation strength. To apply expression (9) to the second order we have to calculate coupling elements for $\Delta m=0$ to the second order, and coupling elements for $|\Delta m|=1$ to the first order. Matrix \bar{M} for $\Delta m=0$ to the second order is given by $d_{\rho\rho} = (\delta^2/2)f'(\rho)^2$, $d_{\rho\theta} = d_{\theta\rho} = 0$, $d_{\theta\theta} = (\delta^2/2)f'(\rho)^2$, $d_{zz} = -(\delta^2/2)f'(\rho)^2$. Matrix \bar{M} for $|\Delta m|=1$ to the first order is given by $d_{\rho\rho} = -(\delta/2)f'(\rho)$, $d_{\rho\theta} = d_{\theta\rho} = \text{sgn}(\Delta m) i (\delta/2)f'(\rho)$, $d_{\theta\theta} = (\delta/2)f'(\rho)$, $d_{zz} = (\delta/2)f'(\rho)$. For $\delta=0.1a$ perturbation we plot (Fig. 4) change in the propagation constant $\Delta\beta$ of the fundamental $m=0$ mode calculated by second order perturbation theory (9) using coupling elements (28). We compare our theoretical predictions with the results of a frequency domain code

called MPB [20], developed at MIT. Over a large range of frequencies we have a good agreement between the second order perturbation theory results and the results of the frequency domain MPB code. Moreover, the rate of convergence of the second order perturbation theory with the number of modes in the expansion basis is almost cubic.

VIII. PMD OF HOLLOW CORE BRAGG FIBER

As a nontrivial application of our method we calculate deterministic PMD due to elliptic perturbation in the high-index-contrast hollow Bragg fiber. Bragg fiber under study is the same as in Ref. [1]. It has a hollow core (index of refraction unity) of $30a$ surrounded by omnidirectional mirror that comprises 17 layers, starting with high-index layer, with indices 4.6/1.6 and thicknesses $0.22a/0.78a$, respectively. Mirror is surrounded by cladding of index 1.6. The frequency of the fundamental $m=1$ mode is chosen at the point of its lowest radiation loss (24 dB/km) in the first omnidirectional band gap and is equal to $\omega=0.24(2\pi c/a)$. If we make this frequency to correspond to the standard $\lambda=1.55\ \mu\text{m}$ of telecommunications, we have $a=\lambda\omega=0.37\ \mu\text{m}$. Note that in the current example the ratio of the structure size (outside core radius) to the size of the smallest feature (mirror layer thickness) is so large that evaluation of PMD using generic frequency domain mode solvers or time domain codes becomes prohibitively computationally intensive. We evaluate PMD in terms of the intermode dispersion parameter τ given by expression (13) using first order perturbation theory that we developed in this paper. We assume the same amount of ellipticity δ for all the layers in the mirror and metallic boundary conditions outside of the last layer of the mirror. Coordinate mapping to describe such an ellipticity will be (1)-(2) with $f(\rho)=\rho$. Coupling elements are calculated to the first order from Eq. (28) with selection rule $|\Delta m|=2$ and \bar{M} matrix given by $d_{\rho\rho}=-\delta$, $d_{\rho\theta}=d_{\theta\rho}=\text{sgn}(\Delta m)i\delta$, $d_{\theta\theta}=\delta$, $d_{zz}=0$. For $\delta=1\%$ ellipticity we plot (Fig. 5) intermode dispersion parameter τ as defined by Eq. (13) for the hollow core Bragg fiber. Note that for the regular telecom-

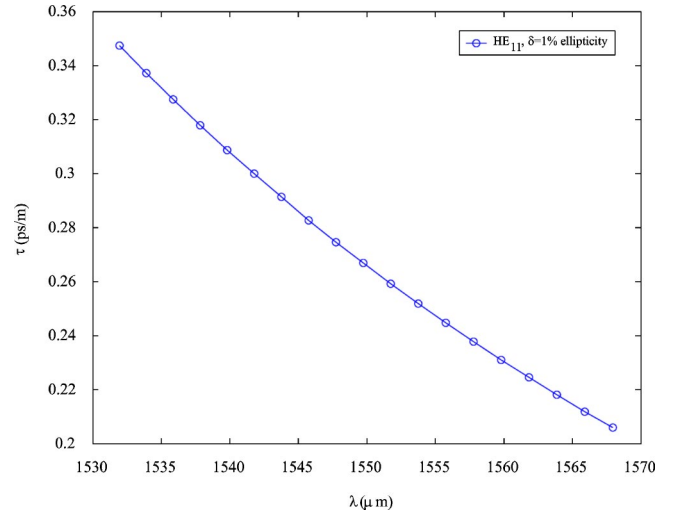


FIG. 5. Intermode dispersion parameter τ for the fundamental HE_{11} mode of a hollow Bragg fiber under the uniform $\delta=1\%$ elliptical perturbation.

munication silica fiber with ellipticity correlation length of 1 m (spun fiber) the intermode dispersion parameter is approximately 0.1 ps/m.

IX. CONCLUSION

In this work, we presented a different form of the coupled mode and perturbation theories to treat general geometric perturbations of a waveguide profile with an arbitrary dielectric index contrast. We demonstrated that for perturbations in high-index-contrast profiles our theories exhibit higher than linear convergence to the exact result with the number of expansion modes, and when perturbation is small perturbation theory approach is demonstrated to give correct results not only to the first order but also to the higher orders of accuracy. To our knowledge, this is the first time perturbation theory approach is developed to deal with arbitrary profile variations in high-index-contrast waveguides to higher orders of accuracy.

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